Chapter 10

Miscellaneous topics

(Lectures 35 and 36)

Keywords : Stability after stall-autorotation and spin ; stability with automatic control; response; transfer function.

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Chapter 10

Miscellaneous topics - 1

Lecture 35

Topics

10.1 Introduction

       To conclude this introductory course on airplane stability and control, the topics like stability after stall, automatic control, Laplace transforms, response and transfer functions are briefly touched upon in this chapter.

10.2 Stability after stall, autorotation and spin

       The airplane is said to have stalled when the angle of attack exceeds the value corresponding to C_{Lmax}. The important effects after stall are as follows.

a) The lift coefficient (C_L) decreases with increase of angle of attack (α) or the slope of the lift curve becomes negative (point C in Fig.10.1).
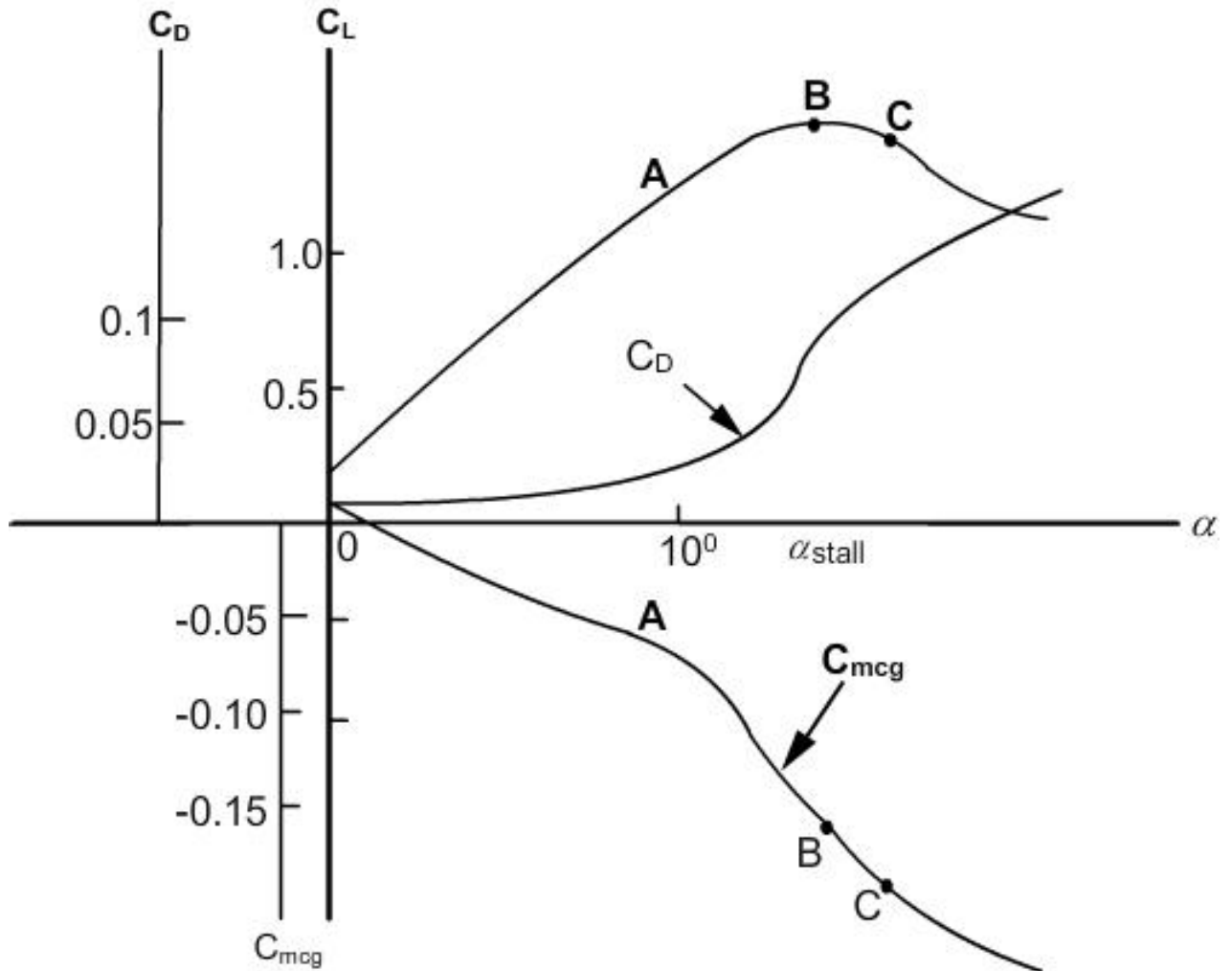
Fig.10.1 Schematic variations of C_L, C_D and C_{mcg} with angle of attack

b) The drag coefficient increases rapidly.

c) The moment coefficient becomes more negative. Compare points A,B and C in Fig.10.1.

  The effects of stall are felt markedly on the lateral stability. As the slope of the lift curve becomes negative, the rolling moment which is of the nature of

damping when $\alpha < \alpha_{stall}$, reverses its sign and becomes a reinforcing effect. Consequently, the airplane would tend to rotate about x-axis. This condition is called autorotation. Later, the airplane may go into a spin. In this flight (spin) the airplane moves along a helical path and rotates about a vertical axis (Fig.10.2). As $\alpha > \alpha_{stall}$, the aileron and elevator have lost their effectiveness. Rudder is the only control effective in spin recovery. To come out of spin the pilot applies appropriate rudder deflection so that the airplane stops spinning. Subsequently, the pilot goes into a dive and pulls-out. An uncontrolled spin would lead to disastrous consequences. Guidelines for design of rudder and its location for spin recovery are available in Ref. 10.1, chapter 4. Some airplanes have a stall warning device which alerts the pilot of approaching stall and to take preventive measures. In a spin-proof airplane the elevator is incapable of achieving C_{Lmax} ; thereby preventing stall and subsequent spin. For further details on autorotation and spin see Ref.1.12, chapters 7 and 8.

Remark:

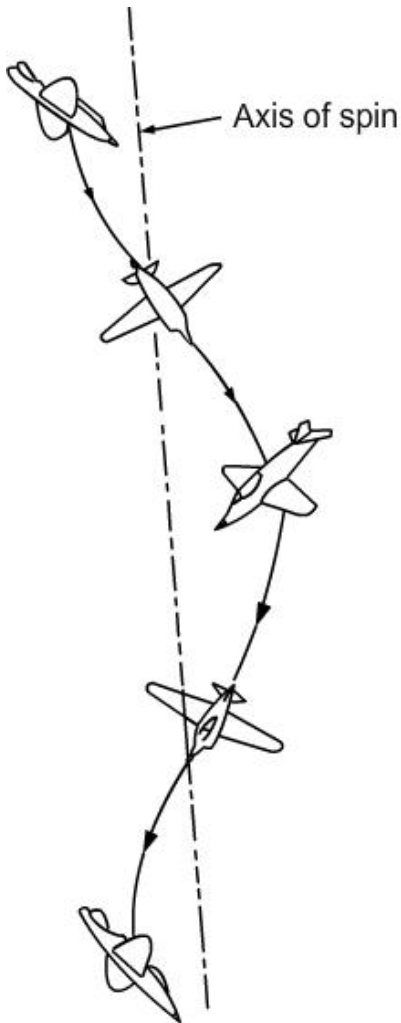For a video clip on motion of Hunter jet in spin, see www.youtube.com.

Fig.10.2 Schematic of flight in a spin

(Adapted from Ref.1.12, chapter 7 with permission from American Institute of
Aeronautics and Astronautics, Inc.)

10.3 Stability with automatic control

In an airplane with automatic control, the control surfaces are deflected
depending on the linear and angular motions of the airplane. This requires the
following equipments.

 a) Appropriate sensors to measure the motion of the airplane.
 b) A flight control computer to calculate the response and the control
    deflections required.
 c) Mechanisms to deflect the controls.

It may be pointed out that the velocity and altitude of the airplane are obtained from the static pressure and total pressure sensed by the Pitot static system. The acceleration is sensed by the accelerometer. The angular positions of the airplane are sensed with the help of the gyros.

After sensing these parameters, the deviations from the desired flight path can be computed by the flight computer. Similarly, the changes over a period of time can be obtained by integrating the variation of the desired parameter over a chosen interval of time. The flight computer also has in its memory the information about the airplane dynamics like values of stability derivatives. The computer calculates the response of the airplane and the control deflections needed to maintain the desired flight path. The computer also sends commands for appropriate deflection of control surfaces.

To illustrate the effect of automatic control on stability of the airplane, consider an elevator which is deflected automatically in response to the changes in (a)velocity (Δu), (b) angle of attack ($\Delta \alpha$) and (c) pitch rate (Δq) i.e.

$$\Delta \delta_e = k_1\ \Delta u + k_2\ \Delta \alpha + k_3\ \Delta q \tag{10.1}$$

The quantities k_1, k_2, k_3 are called gearing ratios or feedback gain and depend on the design of the automatic control system.

To examine the stability of an airplane without automatic control, $\Delta \delta_e$ and $\Delta \delta_T$ were taken as zero in Eq.(8.40). On the other hand, when an airplane has the automatic control, the quantities $\Delta \delta_e$ and $\Delta \delta_T$ would be decided by the values of Δu , $\Delta \alpha$ and Δq. Consequently, equations of motion will change as the terms $X_{\delta e}\, \Delta \delta_e$, $Z_{\delta e}\, \Delta \delta_e$,$M_{\delta e}\, \Delta \delta_e$ have values depending on the gearing ratios. Hence, in the state space form of equation:

$$\dot{\mathbf{X}} = \mathbf{A.\ X + B.\ \eta},$$

the vector $\boldsymbol{\eta}$ becomes:

$$\boldsymbol{\eta} = -\mathbf{k}^{\mathrm{T}}\ \mathbf{X} + \boldsymbol{\eta}' \tag{10.2}$$

where $\mathbf{k}^{\mathrm{T}}$ is the transpose of the feedback gain vector and $\boldsymbol{\eta}'$ is the pilot input. Thus, for an airplane with automatic control :

$$\dot{\mathbf{X}} = (\mathbf{A - Bk}^{\mathrm{T}})\ \mathbf{X} + \mathbf{B}\ \boldsymbol{\eta}' \tag{10.3}$$

Or $\dot{\mathbf{X}} = \mathbf{A}^* \mathbf{X} + \mathbf{B} \; \boldsymbol{\eta}'$, where, $\mathbf{A}^* = (\mathbf{A} - \mathbf{B}\mathbf{k}^T)$ (10.4)

A* is called augmented matrix.

The characteristic equation for Eq.(10.3) would be:

$| \lambda_r \; \mathbf{I} - \mathbf{A}^* | = 0$ (10.5)

Expanding Eq.(10.5) gives the following new stability quartic:

$A_2 \; \lambda^4 + B_2 \; \lambda^3 + C_2 \; \lambda^2 + D_2 \; \lambda + E_2 = 0$ (10.6)

where the coefficients A_2, B_2, C_2, D_2 and E_2 involve k_1, k_2 and k_3 in addition to the stability derivatives of the airplane. Thus, it is noticed that by automatic control the coefficients of the characteristic equation get modified. Hence, the stability of the airplane is altered. This can be explained in other words as follows.

Recall that the stability of an airplane is its inherent ability to return to the equilibrium position after the disturbance. Now, for an airplane with automatic control, as the name implies, the changes in the control deflections are brought about without pilot's intervention. Hence, the inherent features of the airplane and consequently the stability levels are changed by automatic control.

Yaw damper

It was pointed out in section 9.8 that in lateral dynamic stability under certain situations, either the spiral mode or the Dutch roll mode would be unstable. This situation can be corrected with the use of a yaw damper which is a simple form of automatic control to change the rudder deflection based on rate of yaw (see Ref.1.1, chapter 5 for details).

10.3.2 Variable stability airplane

In some airplanes it is possible to change on ground, the feedback gains k_1, k_2, k_3 etc. As a consequence of this change, the stability of the airplane can be altered. An airplane with such provisions is called a variable stability airplane. Such airplanes were built in the past and were used to study the flying qualities of airplanes.

Remark:

Fly-by- wire, mentioned in section 6.12, and used on current airplanes, is a later development of automatic control.

10.4 Response

To study the response of an airplane, it is helpful to know the mathematical technique called Laplace transform. This is discussed in the next subsection.

10.4.1 Laplace transform

By using Laplace transform, a set of linear differential equations can be converted into a set of algebraic equations. This set of algebraic equations can be easily solved and the solution of the differential equations is obtained by taking the inverse transform. A brief outline of the technique is presented here. The description is based on Ref.1.1, Appendix 'C'. Standard books on mathematics can be consulted for further details.

Laplace transform is a mathematical operation defined by:

$$L\left[f(t)\right] = \int_{0}^{\infty} f(t)\, e^{-st}\, dt\ = F(s) \tag{10.7}$$

Here, f(t) is a function of time, L is the Laplace operator, s is a complex variable and F(s) is the Laplace transform of f(t).

The inverse Laplace transform is denoted by:

$$f\,(t) = L^{-1}[F(s)] \tag{10.8}$$

Some examples are presented below:

(i) $f(t) = e^{-a\,t}$

$$L\left[e^{-at}\right] = \int_{0}^{\infty} e^{-a\,t}\ e^{-s\,t}\ dt = \int_{0}^{\infty} e^{-(a+s)\,t}\ dt = \left[\frac{e^{-(a+s)\,t}}{a+s}\right]_{0}^{\infty} = \frac{1}{s+a}$$

$$\text{i.e.,}\,L\left[e^{-a\,t}\right] = F\,(s) = \frac{1}{s+a} \tag{10.9}$$

(ii)   Note: When a = 0   Eq.(10.9) gives :

$$L[1] = \frac{1}{s} \tag{10.10}$$

(iii) f(t) = sin ωt

$$L[\sin \omega t] = \int_0^\infty e^{-st}\sin \omega t \, dt = \frac{1}{2i}\int_0^\infty (e^{i\omega t}-e^{-i\omega t})e^{-st} \, dt = \frac{\omega}{s^2+\omega^2}$$

i.e., $L[\sin \omega t] = F(s) = \dfrac{\omega}{s^2 + \omega^2}$ (10.11)

iv) In a similar manner

$$L\left[\cos \omega t\right] = \frac{s}{s^2 + \omega^2}$$ (10.12)

v) $f(t) = dy / dt$

$$L\left[\frac{dy}{dt}\right] = \int_0^\infty \frac{dy}{dt} e^{-st} \, dt = \left[y \, e^{-st}\right]_0^\infty + s \int_0^\infty y \, e^{-st} \, dt$$

But, $\int_0^\infty y \, e^{-st} \, dt = L[y(t)] = Y(s)$

Hence, $L\left[\dfrac{dy}{dt}\right] = -y(0) + s \, Y(s)$ (10.13)

vi) In a similar manner,

$$L\left[\frac{d^n y}{d t^n}\right] = s^n \, Y(s) - s^{n-1} \, y(0) - s^{n-2} \left[\frac{dy}{dt}\right]_{t=0} - \dots - \left[\frac{d^{n-1} y}{d t^{n-1}}\right]_{t=0}$$ (10.14)

if $y(0) = \left[\dfrac{dy}{dt}\right]_{t=0} = \dots = \left[\dfrac{d^{n-1} y}{d t^{n-1}}\right]_{t=0} = 0$ then

$$L\left[\frac{d^n y}{d t^n}\right] = s^n \, Y(s)$$ (10.15)

vii) $f(t) = \int_0^t y(\tau) d\tau$

Then $L\left[\int_0^t y(\tau) \, d\tau\right] = \int_0^\infty e^{-st} \int_0^t y(\tau) \, d\tau \, dt$

Integrating by parts, gives the following final result:

$$L\left[\int_0^t y(\tau) \, d\tau\right] = \frac{Y(s)}{s}$$ (10.16)

Remarks:

i) Appendix 'C' of Ref.1.1 gives F(s) for some more functions.

ii) See Appendix 'C' of Ref 1.1 and Ch. 6 of Ref 10.2 for use of Laplace transform to solve ordinary differential equations. In the next subsection the use of Laplace transform to obtain response of a two degrees of freedom system is illustrated.

10.4.2 Response of two degrees of freedom system to initial disturbance – solutions by classical method and by Laplace transform

A simple case is considered to illustrate as to how response can be obtained for initial disturbance. The Two methods, namely, the classical method and Laplace transform technique are used.

In the classical method, the equations subject to initial conditions are solved. It will be evident later that this method becomes complex when control deflections are involved.

The two methods are now explained with an example*.

Consider the following system with two degrees of freedom.

$$3\dot{x}_1 + 2x_1 + \dot{x}_2 = 0 \tag{10.17}$$

$$\dot{x}_1 + 4\dot{x}_2 + 3x_2 = 0 \tag{10.18}$$

The initial conditions for the stability type problem are taken as: $x_1(0) = 1$, $x_2(0) = 0$.

Remark:

In stability problems we take disturbance to be small. Here, $x_1(0)$ is taken as unity for simplicity. It could also have been taken 0.1or 0.01.

Solution by the classical method:

Let the solutions be $x_1 = \rho_1 e^{\lambda t}$ and $x_2 = \rho_2 e^{\lambda t}$. Substituting these in Eqs.(10.17) and (10.18), gives:

$$3\lambda\rho_1 + 2\rho_1 + \lambda\rho_2 = 0 \tag{10.19}$$

$$\lambda\rho_1 + 4\lambda\rho_2 + 3\rho_2 = 0 \tag{10.20}$$

For a non-trivial solution of Eqs.(10.19) & (10.20) to exist, :

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*Taken from section 7 chapter 6 of Ref.8.3 with permission from McGraw-Hill Book Company.

$$\begin{vmatrix} 3\lambda + 2 & \lambda \\ \lambda & 4\lambda + 3 \end{vmatrix} = 0 \tag{10.21}$$

This simplifies to $11\lambda^2 + 17\lambda + 6 = 0$. Which gives the roots as: $\lambda = -1$ and $\lambda = -6/11$.

Thus, the general solution to the chosen system is written as:

$$x_1 = A_1\, e^{-t} + A_2\, e^{-\frac{6}{11}t} \tag{10.22}$$

$$x_2 = B_1\, e^{-t} + B_2\, e^{-\frac{6}{11}t} \tag{10.23}$$

To get the response, following the disturbance, A_1, A_2, B_1 and B_2 should be evaluated. Using initial conditions the following two relations are obtained.

$$1 = A_1\, e^0 + A_2\, e^0 \text{ or } A_2 = 1 - A_1 \tag{10.24}$$

$$\text{and} \quad 0 = B_1\, e^0 + B_2\, e^0 \text{ or } B_1 = -B_2 \tag{10.25}$$

To get two more relationships, the following arguments can be applied.

a) Since, $x_1 = A_1\, e^{-t}$, and $x_2 = B_1\, e^{-t}$ are the solutions of the governing equations, substituting these in Eq.(10.17) gives:

$$- 3 A_1\, e^{-t} + 2 A_1\, e^{-t} - B_1\, e^{-t} = 0$$

Or: $A_1 = -B_1$. \hfill (10.26)

b) Since, $x_1 = A_2\, e^{-6t/11}$, and $x_2 = B_2\, e^{-6t/11}$ are also the solutions of the governing equations, substituting these in Eq.(10.17) gives:

$$3(-6/11)A_2\, e^{-6t/11} + 2A_2\, e^{-6t/11} + (-6/11)\, B_2 e^{-6t/11} = 0$$

Or: $A_2 = (3/2)B_2$ \hfill (10.27)

Equations (10.24) to (10.27) are the four equations for the constants A_1, A_2, B_1 and B_2. Solving these equations yields:

$$A_1 = 2/5,\ B_1 = -2/5,\ A_2 = 3/5 \text{ and } B_2 = 2/5 \tag{10.28}$$

Hence, the response of the chosen two degrees of freedom system to the given disturbance is:

$$x_1 = \frac{2}{5}e^{-t} + \frac{3}{5}e^{-\frac{6}{11}t} \tag{10.29}$$

$$x_2 = \frac{2}{5}e^{-\frac{6}{11}t} - \frac{2}{5}e^{-t} \tag{10.30}$$

Remark:

Equation (10.17) was used to obtain the relations given by Eqs.(10.26) and Eq.(10.17). Same relations are obtained when Eq.(10.18) is used.

Solution by Laplace transform:

Now, the set of Eqs.(10.17) and (10.18) is solved using Laplace transform technique.

Let, $L(x_1) = y_1$ and $L(x_2) = y_2$.

From Eq.(10.13) $L(d\,x_1 / dt) = -x_1(0) + s\,y_1$ and $L(d\,x_2 / dt) = -x_2(0) + s\,y_2$

The initial conditions are $x_1(0) = 1$ and $x_2(0) = 0$.

Hence, $L(dx_1 / dt) = -1 + s\,y_1$ and $L(d\,x_2 / dt) = s\,y_2$

It may be noted that while taking Laplace transform the initial conditions get automatically incorporated.

Taking Laplace transform of Eqs.(10.17) and (10.18) gives :

$$3(-1 + y_1 s) + 2\,y_1 + s\,y_2 = 0 \tag{10.31}$$

$$(-1 + y_1 s) + 4sy_2 + 3y_2 = 0 \tag{10.32}$$

Solving, Eqs.(10.31) and (10.32) yields:

$$y_1 = \frac{(11s+9)}{(s+1)(11s+6)}, \tag{10.33}$$

$$y_2 = \frac{2}{(s+1)(11s+6)} \tag{10.34}$$

Before taking the inverse transform of Eqs.(10.33) and (10.34), they are simplified by using partial fractions.

It may be noted that :

$$\frac{1}{(s+a)(s+b)} = \frac{1}{(b-a)}\left\{\frac{1}{(s+a)} - \frac{1}{(s+b)}\right\} \tag{10.35}$$

$$\frac{(s+c)}{(s+a)(s+b)} = \frac{(c-a)}{(b-a)(s+a)} - \frac{(c-b)}{(b-a)(s+b)} \tag{10.36}$$

Hence, $L^{-1}\left\{\dfrac{1}{(s+a)(s+b)}\right\} = \dfrac{e^{-bt} - e^{-at}}{a-b}$

And $L^{-1}\{\dfrac{(s+c)}{(s+a)(s+b)}\} = \dfrac{(c-a)\,e^{-at} - (c-b)\,e^{-bt}}{b-a}$

Applying these to Eqs.(10.33) and (10.34) and simplifying gives :

$$x_1 = \frac{2}{5}\,e^{-t} + \frac{3}{5}\,e^{-\frac{6}{11}t} \quad \text{and} \quad x_2 = \frac{2}{5}\,e^{-\frac{6}{11}t} - \frac{2}{5}\,e^{-t}$$

which are the same as the solution obtained by the classical method viz. Eq.(10.29) and (10.30).

Remarks:

    i)The two degrees of freedom system chosen here, has two modes corresponding each root. Since, both the roots are negative, the system is stable. Further, the roots are real and hence the response is an aperiodic motion.

    ii)The motions corresponding to the two modes and their sum are plotted in Figs.10.3 and 10.4. It is observed that at t = 0, x_1 equals one and x_2 equals zero, which satisfy the prescribed initial conditions.

    iii)The first mode, corresponding to e^{-t} , has higher damping than the second mode, corresponding to $e^{-6t/11}$. Hence, the first mode goes to zero faster than the second mode. However, both the modes go to zero in course of time as the system is stable.
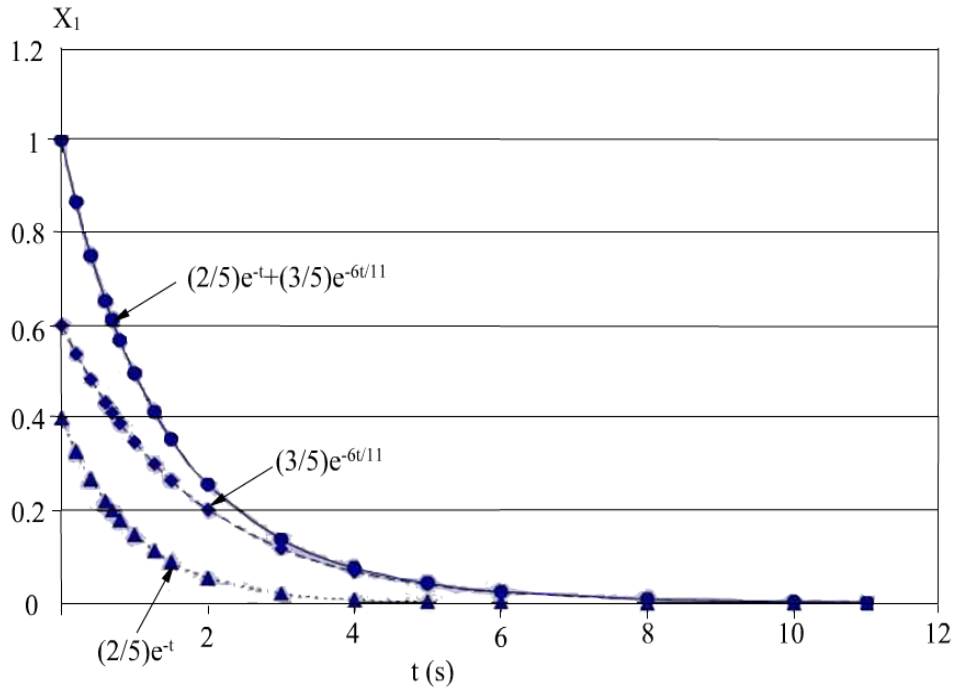
Fig.10.3 Response of a two degrees of freedom system –
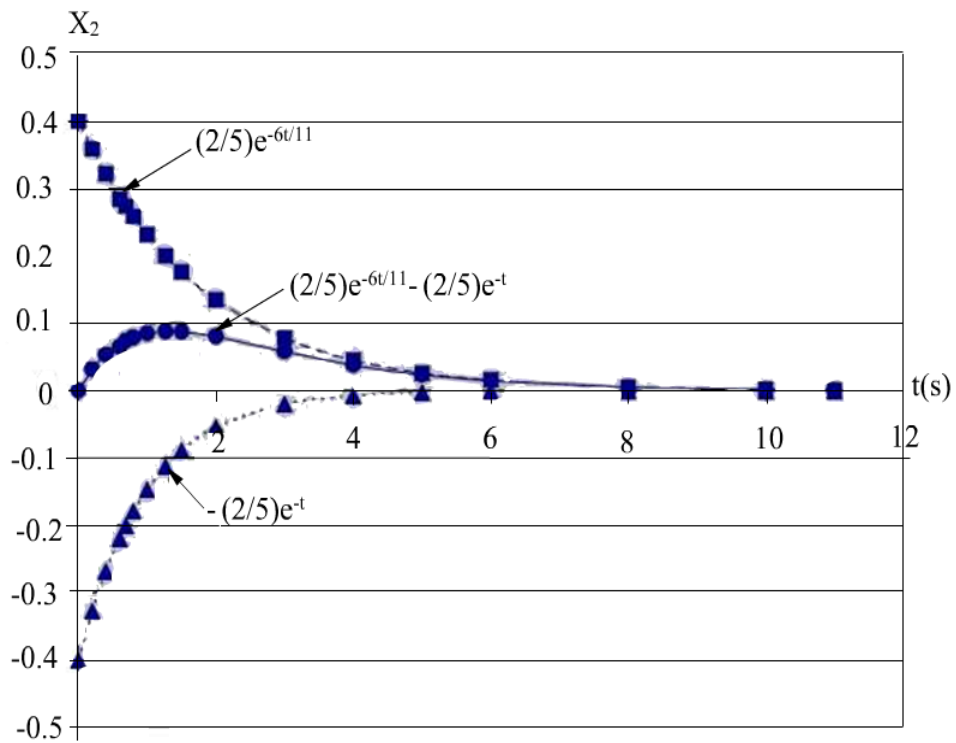variation of x_1 with time

Fig.10.4 Response of a two degrees of freedom system – variation of x_2 with
time

10.4.3 Response of the two degrees of freedom system to specified input

Consider a sudden input of unit magnitude at $t = 0$ which subsequently remains constant (Fig.10.5a) . The equations of motion in this case become:

$$3\dot{x}_1 + 2x_1 + \dot{x}_2 = 1, t \geq 0 \tag{10.37}$$

$$\dot{x}_1 + 4\dot{x}_2 + 3x_2 = 0 \tag{10.38}$$

The initial conditions are: $x_1 = 0$, $x_2 = 0$ at $t = 0$. The solution conveniently obtained using the Laplace transform technique. Taking the Laplace transform of Eqs.(10.37) and (10.38) yields:

$$(3s + 2) y_1 + s y_2 = \frac{1}{s} \tag{10.39}$$

$$s y_1 + (4s + 3) y_2 = 0 \tag{10.40}$$

Solving Eqs.(10.39) and (10.40) gives:

$$y_1 = \frac{4s + 3}{s(s + 1)(11s + 6)}, \tag{10.41}$$

$$y_2 = \frac{-1}{(s + 1)(11s + 6)} \tag{10.42}$$

Using Eqs.(10.35) and (10.36) and taking the inverse Laplace transform, results in the following solution.

$$X_1 = \frac{1}{2} - \frac{1}{5}e^{-t} - \frac{3}{10}e^{-\frac{6}{11}t} \tag{10.43}$$

$$X_2 = \frac{1}{5}(e^{-t} - e^{-\frac{6}{11}t}) \tag{10.44}$$

Remarks:

i)The variations of x_1 and x_2 with time are shown in Figs.10.5b and c. It is observed that after some time the terms $e^{-6t/11}$ and e^{-t} tend to zero. The steady state values of x_1 and x_2 are 1/2 and 0 respectively. Thus, a control input of one unit results in an output of half unit for this system.

ii)It is interesting to note that the steady state values of x_1 and x_2 can be obtained from the original equations (Eqs.10.37 and 10.38) by noting that in steady state all the derivatives become zero. Thus, in steady state the left hand side of Eq.(10.37) is $2x_1$ and r.h.s equals one giving $x_1 = (1/2)$. Similarly, Eq.(10.38) gives that $x_2 = 0$ in steady state.

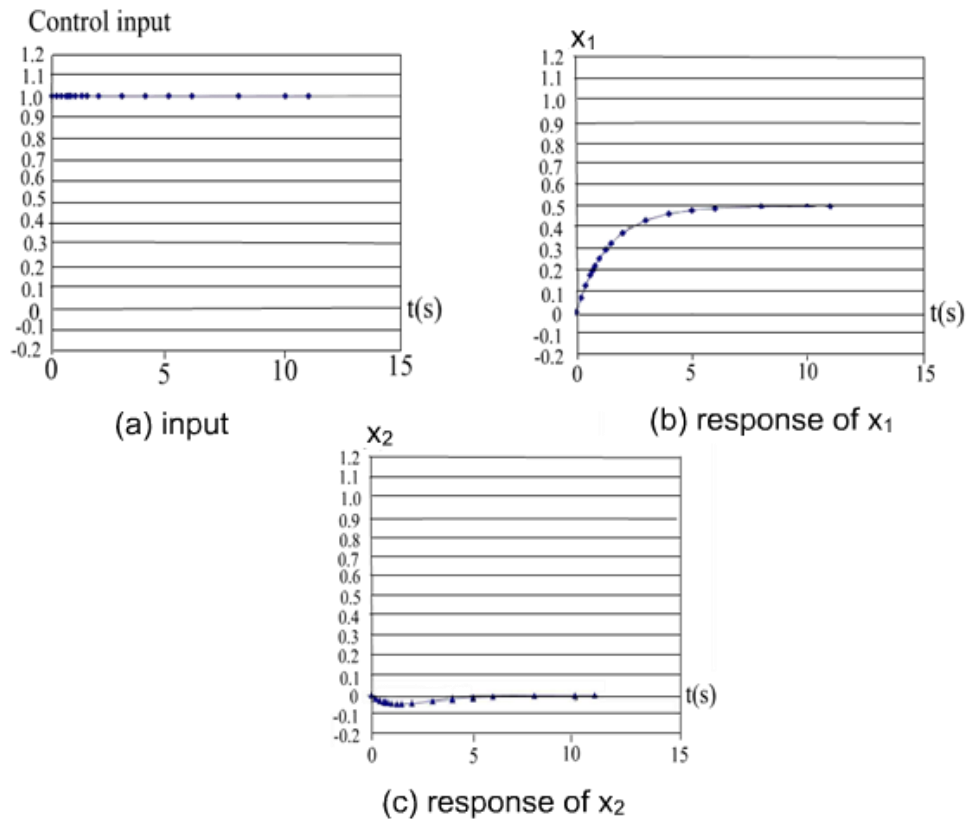(a) input

(b) response of x_1

(c) response of x_2

Fig.10.5 Response of a two degrees of freedom system to specified input